

Maximum norm stability and error estimates for the evolving surface finite element method

Balázs Kovács* and Christian Andreas Power Guerra*

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Abstract

We show convergence in the natural L^∞ - and $W^{1,\infty}$ -norm for a semidiscretization with linear finite elements of a linear parabolic partial differential equations on evolving surfaces. To prove this we show error estimates for a Ritz map, error estimates for the material derivative of a Ritz map and a weak discrete maximum principle.

Keywords: evolving surfaces, parabolic PDE, ESFEM, weighted norm, weak discrete maximum principle;

1 Introduction

Many important problems can be modeled by partial differential equations (PDEs) on evolving surfaces. Examples for such equations are given in material sciences, fluid mechanics and biophysics [16, 18, 14]. The basic linear parabolic PDE on a moving surface is

$$\partial^\bullet u + u \nabla_{\Gamma(t)} \cdot v - \Delta_{\Gamma(t)} u = f \quad \text{on } \Gamma(t).$$

Here the velocity v is explicitly given and we seek to compute a numerical approximation to the exact solution u . Dziuk and Elliott [10] introduced the evolving surface finite element method (ESFEM) to solve this problem. Error estimates for the semidiscretization with piecewise linear finite elements in the L^2 - and H^1 -norm are given in [12, 9].

The aim of this work is to give error bounds for the semidiscretization with linear finite elements in the L^∞ - and $W^{1,\infty}$ -norm. Such estimates are of interest for nonlinear parabolic PDEs on evolving surfaces and if the velocity v is not explicitly given, but depends on the exact solution u . Example of such problems are given in [5, 16, 2, 15, 3] and the references therein. The treatment of such more general equations are beyond the scope of this paper.

Our convergence proof for the semidiscretization of the linear heat equation on evolving surfaces relies on three main results. In our first result we give some

*Mathematisches Institut, University of Tübingen, Auf der Morgenstelle 10, 72076 Tübingen, Germany, E-mail address: {kovacs,power}@na.uni-tuebingen.de

error bounds in the L^∞ - and $W^{1,\infty}$ -norm for a suitable Ritz map. We give a proof based on Nitsche's weighted norm technique [24].

Our second result gives bounds in the L^∞ - and $W^{1,\infty}$ -norm for our material derivative Ritz map error, since the time derivative does not commute with our Ritz map.

Our third result extends a weak finite element maximum principle, which is due to Schatz, Thomée and Wahlbin [25], to the evolving surface case. In [25] they use basic properties of the semigroup corresponding to the linear heat equation on a bounded domain. Since there is no semigroup theory for the linear heat equation on evolving surfaces we are going to use a different approach.

The layout of the paper is as follows. We begin in Section 2 by fixing some notation and introducing the most basic notion. In the first three subsection of Section 3 we quickly develop the evolving surface finite element method (ES-FEM) and recall basic results and estimates. In the following three subsection we introduce a surface version of Nitsche's weighted norms and finish with an L^2 -projection. In Section 4 we give error bound in the maximum norm for our Ritz map. In Section 5 we derive a weak ESFEM maximum principle. In Section 6 we give error bounds for the semi discretization of the linear heat equation on evolving surfaces in the L^∞ - and $W^{1,\infty}$ -norm. In Section 7 we present the results of a numerical experiment. We gather technical details for calculations with our weight functions in Appendix B.

2 A parabolic problem on evolving surfaces

Let us consider a smooth evolving closed hypersurface $\Gamma(t) \subset \mathbb{R}^{m+1}$ (our main focus is on the case $m = 2$, but some of our results hold for more general cases), $0 \leq t \leq T$, which moves with a given smooth velocity v . More precise we assume that there exists a smooth dynamical system $\Phi: \Gamma_0 \times [0, T] \rightarrow \mathbb{R}^{m+1}$, such that for each $t \in [0, T]$ the map $\Phi_t := \Phi(\cdot, t)$ is an embedding. We define $\Gamma(t) := \Phi_t(\Gamma_0)$ and define the velocity v via the equation $\partial_t \Phi(x, t) = v(\Phi(x, t), t)$. Let $\partial^\bullet u = \partial_t u + v \cdot \nabla u$ denote the material derivative of the function u . The tangential gradient is given by $\nabla_\Gamma u = \nabla u - \nabla u \cdot \nu \nu$, where ν is the unit normal and finally we define the Laplace–Beltrami operator via $\Delta_\Gamma u = \nabla_\Gamma \cdot \nabla_\Gamma u$. This article shares the setting of Dziuk and Elliott [9, 12], and [22].

We consider the following linear problem derived in [9, Section 3]:

$$\begin{cases} \partial^\bullet u + u \nabla_{\Gamma(t)} \cdot v - \Delta_{\Gamma(t)} u = f & \text{on } \Gamma(t), \\ u(\cdot, 0) = u_0 & \text{on } \Gamma(0). \end{cases} \quad (1) \text{ ?eq_ES-PDE-strong-form?}$$

We use Sobolev spaces on surfaces: For a sufficiently smooth surface Γ and $1 \leq p \leq \infty$ we define

$$W^{1,p}(\Gamma) = \{\eta \in L^p(\Gamma) \mid \nabla_\Gamma \eta \in L^p(\Gamma)^{m+1}\},$$

and analogously $W^{k,p}(\Gamma)$ for $k \in \mathbb{N}$ [9, Section 2.1]. We set $H^k(\Gamma) = W^{k,2}(\Gamma)$. Finally, \mathcal{G}_T denotes the space-time manifold, i.e. $\mathcal{G}_T := \cup_{t \in [0, T]} \Gamma(t) \times \{t\}$.

If $f = 0$ then a weak formulation of this problem reads as follows.

Definition 2.1 (weak solution, [9] Definition 4.1). *A function $u \in H^1(\mathcal{G}_T)$ is called a weak solution of (1), if for almost every $t \in [0, T]$*

$$\frac{d}{dt} \int_{\Gamma(t)} u \varphi + \int_{\Gamma(t)} \nabla_{\Gamma(t)} u \cdot \nabla_{\Gamma(t)} \varphi = \int_{\Gamma(t)} u \partial^\bullet \varphi$$

holds for every $\varphi \in H^1(\mathcal{G}_T)$ and $u(\cdot, 0) = u_0$.

For suitable f and u_0 existence and uniqueness results, for the strong and the weak problem, were obtained in [9, Section 4].

Throughout this article we assume that f and u_0 are such regular that $u \in W^{3,\infty}(\mathcal{G}_T)$. Furthermore we set for simplicity reasons in all sections $f = 0$, since the extension of our results to the inhomogeneous case are straightforward.

3 Preliminaries

We give a summary of this section. In Section 3.1 we introduce the ESFEM, which is due to Dziuk and Elliott [9]. In Section 3.2 we recall the lifting process, which originates in Dziuk [8]. In Section 3.3 we collect important results from Dziuk and Elliott [12] and sometimes state them in a slightly more general fashion. In Section 3.4 we introduce weighted norms, which are due to Nitsche [24], and give connections to the L^∞ -norm. In Section 3.5 we give interpolation estimates in the L^2 -, L^∞ - and weighted norms and further give some special interpolation estimates in weighted norms. The latter two were first stated in Nitsche [24]. In Section 3.6 we introduce an L^2 -projection, give a stability bound in L^p -norms and finish with a error estimate with respect to a different weight function. The basic reference for this is Douglas, Dupont, Wahlbin [7] and Schatz, Thomée, Wahlbin [25].

3.1 Semidiscretization with the evolving surface finite element method

The smooth surface $\Gamma(t)$ is approximated by a triangulated one denoted by $\Gamma_h(t)$, whose vertices $a_j(t) = \Phi(a_j(0), t)$ are sitting on the surface for all time, such that

$$\Gamma_h(t) = \bigcup_{E(t) \in \mathcal{T}_h(t)} E(t).$$

We always assume that the (evolving) simplices $E(t)$ are forming an admissible triangulation $\mathcal{T}_h(t)$, with h denoting the maximum diameter. Admissible triangulations were introduced in [9, Section 5.1]: Every $E(t) \in \mathcal{T}_h(t)$ satisfies that the inner radius σ_h is bounded from below by ch with $c > 0$, and $\Gamma_h(t)$ is not a global double covering of $\Gamma(t)$. The discrete tangential gradient on the discrete surface $\Gamma_h(t)$ is given by

$$\nabla_{\Gamma_h(t)} \phi := \nabla \phi - \nabla \phi \cdot \nu_h \nu_h,$$

understood in a piecewise sense, with ν_h denoting the normal to $\Gamma_h(t)$ (see [9]).

For every $t \in [0, T]$ we define the finite element subspace $S_h(t)$ spanned by the continuous, piecewise linear evolving basis functions χ_j , satisfying

$$\chi_j(a_i(t), t) = \delta_{ij} \quad \text{for all } i, j = 1, 2, \dots, N,$$

therefore $S_h(t) = \text{span}\{\chi_1(\cdot, t), \chi_2(\cdot, t), \dots, \chi_N(\cdot, t)\}$.

We interpolate the dynamical system Φ by $\Phi_h: \Gamma_h(0) \rightarrow \mathbb{R}^{m+1}$, the discrete dynamical system of $\Gamma_h(t)$. This defines a discrete surface velocity V_h via $\partial_t \Phi_h(y_h, t) = V_h(\Phi_h(y_h, t), t)$. Then the discrete material derivative is given by

$$\partial_h^\bullet \phi_h = \partial_t \phi_h + V_h \cdot \nabla \phi_h \quad (\phi_h \in S_h(t)).$$

The key *transport property* derived in [9, Proposition 5.4], is the following

$$\partial_h^\bullet \chi_k = 0 \quad \text{for } k = 1, 2, \dots, N. \quad (2) \text{ ?eq: transport property?}$$

The spatially discrete problem for evolving surfaces is: Find $U_h \in S_h(t)$ such that

$$\frac{d}{dt} \int_{\Gamma_h(t)} U_h \phi_h + \int_{\Gamma_h(t)} \nabla_{\Gamma_h} U_h \cdot \nabla_{\Gamma_h} \phi_h = \int_{\Gamma_h(t)} U_h \partial_h^\bullet \phi_h, \quad (\forall \phi_h \in S_h(t)), \quad (3) \text{ ?eq: semidiscrete problem?}$$

with the initial condition $U_h(\cdot, 0) = U_h^0 \in S_h(0)$ being a sufficient approximation to u_0 .

3.2 Lifts

In the following we recall the so called *lift operator*, which was introduced in [8] and further investigated in [9, 12]. The lift operator projects a finite element function on the discrete surface onto a function on the smooth surface.

Using the *oriented distance function* d ([9, Section 2.1]), for a continuous function $\eta_h: \Gamma_h(t) \rightarrow \mathbb{R}$ its lift is define as

$$\eta_h^l(x^l, t) := \eta_h(x, t), \quad x \in \Gamma(t),$$

where for every $x \in \Gamma_h(t)$ the value $x^l = x^l(x, t) \in \Gamma(t)$ is uniquely defined via $x = x^l + \nu(x^l, t)d(x, t)$. This notation for x^l will also be used later on. By η^{-l} we mean the function whose lift is η , and by E_h^l we mean the lift of the triangle E_h .

The following pointwise estimate was shown in the proof of Lemma 3 from Dziuk [8]:

$$\frac{1}{c} |\nabla_{\Gamma} \eta_h^l(x^l)| \leq |\nabla_{\Gamma_h} \eta_h(x)| \leq c |\nabla_{\Gamma} \eta_h^l(x^l)|. \quad (4) \text{ ?eq:66?}$$

We now recall some notions using the lifting process from [8, 9]. We have the lifted finite element space

$$S_h^l(t) := \{\varphi_h = \phi_h^l \mid \phi_h \in S_h(t)\}.$$

By δ_h we denote the quotient between the continuous and discrete surface measures, dA and dA_h , defined as $\delta_h dA_h = dA$. For these quantities we recall some results from [9, Lemma 5.1], [12, Lemma 5.4] and [22, Lemma 6.1].

Lemma 3.1. *We have the estimates*

$$\|d\|_{L^\infty(\Gamma_h(t))} \leq ch^2, \quad \|1 - \delta_h\|_{L^\infty(\Gamma_h(t))} \leq ch^2,$$

with constants independent of t and h .

3.3 Geometric estimates and bilinear forms

Let us denote by $\Phi_h^l: \Gamma_0 \times [0, T] \rightarrow \mathbb{R}^{m+1}$ the lift of Φ_h . We define the velocity v_h via the formula $\partial_t \Phi_h^l(x, t) = v_h(\Phi_h^l(x, t), t)$. Then the discrete material derivative on $\Gamma(t)$ is given by

$$\partial_h^\bullet u = \partial_t u + v_h \cdot \nabla u,$$

which satisfies the following relations, cf. [12]:

$$\partial^\bullet u = \partial_h^\bullet u + (v_h - v) \cdot \nabla_\Gamma u, \quad (5) \text{ ?eq:40?}$$

$$\|v - v_h\|_{L^\infty(\Gamma(t))} + h\|v - v_h\|_{W^\infty(\Gamma(t))} \leq ch^2\|v\|_{W^{2,\infty}(\Gamma(t))}, \quad (6) \text{ ?eq: velocity L infty bound?}$$

We use the time dependent bilinear forms defined in [12, Section 3.3]: for $z, \varphi \in H^1(\Gamma(t))$ and $Z_h, \phi_h \in H^1(\Gamma_h(t))$:

$$\begin{aligned} a(t; z, \varphi) &= \int_{\Gamma(t)} \nabla_\Gamma z \cdot \nabla_\Gamma \varphi, & a_h(t; Z_h, \phi_h) &= \sum_{E \in \mathcal{T}_h} \int_E \nabla_{\Gamma_h} Z_h \cdot \nabla_{\Gamma_h} \phi_h, \\ m(t; z, \varphi) &= \int_{\Gamma(t)} z \varphi, & m_h(t; Z_h, \phi_h) &= \int_{\Gamma_h(t)} Z_h \phi_h \\ g(t; v; z, \varphi) &= \int_{\Gamma(t)} (\nabla_\Gamma \cdot v) z \varphi, & g_h(t; V_h; Z_h, \phi_h) &= \int_{\Gamma_h(t)} (\nabla_{\Gamma_h} \cdot V_h) Z_h \phi_h, \\ b(t; v; z, \varphi) &= \int_{\Gamma(t)} \mathcal{B}(v) \nabla_\Gamma z \cdot \nabla_\Gamma \varphi, \\ b_h(t; V_h; Z_h, \phi_h) &= \sum_{E \in \mathcal{T}_h} \int_E \mathcal{B}_h(V_h) \nabla_{\Gamma_h} Z_h \end{aligned}$$

where the discrete tangential gradients are understood in a piecewise sense, and with the matrices

$$\begin{aligned} \mathcal{B}(v)_{ij} &= \delta_{ij}(\nabla_\Gamma \cdot v) - ((\nabla_\Gamma)_i v_j + (\nabla_\Gamma)_j v_i), \\ \mathcal{B}_h(V_h)_{ij} &= \delta_{ij}(\nabla_{\Gamma_h} \cdot V_h) - ((\nabla_{\Gamma_h})_i (V_h)_j + (\nabla_{\Gamma_h})_j (V_h)_i), \end{aligned}$$

where $i, j = 1, 2, \dots, m+1$.

The time derivatives of the bilinear forms are given in the following lemma.

Lemma 3.2 (Discrete transport property). *For $z, \varphi \in H^1(\Gamma(t))$ we have*

$$\begin{aligned} \frac{d}{dt} m(z, \varphi) &= m(\partial_h^\bullet z, \varphi) + m(z, \partial_h^\bullet \varphi) + g(v_h; z, \varphi), \\ \frac{d}{dt} a(z, \varphi) &= a(\partial_h^\bullet z, \varphi) + a(z, \partial_h^\bullet \varphi) + b(v_h; z, \varphi). \end{aligned} \quad (7) \text{ ?eq: continuous transport prop}$$

Similarly for $Z_h, \phi_h \in H^1(\Gamma_h(t))$ we have

$$\begin{aligned} \frac{d}{dt} m_h(Z_h, \phi_h) &= m_h(\partial^\bullet Z_h, \phi_h) + m_h(Z_h, \partial^\bullet \phi_h) + g_h(V_h; Z_h, \phi_h), \\ \frac{d}{dt} a_h(Z_h, \phi_h) &= a_h(\partial^\bullet Z_h, \phi_h) + a_h(Z_h, \partial^\bullet \phi_h) + b_h(V_h; Z_h, \phi_h). \end{aligned} \quad (8) \text{ ?eq: discrete transport prop}$$

Important and often used results are the bounds of the geometric perturbation errors in the bilinear forms.

Lemma 3.3. *For all $1 \leq p, q \leq \infty$, that are conjugate, $p^{-1} + q^{-1} = 1$, and for arbitrary $Z_h \in L^p(\Gamma_h(t))$ and $\phi_h \in L^q(\Gamma_h(t))$, with corresponding lifts $z_h \in L^p(\Gamma(t))$ and $\varphi_h \in L^q(\Gamma(t))$ we have the following estimates:*

$$\begin{aligned} |m(z_h, \varphi_h) - m_h(Z_h, \phi_h)| &\leq ch^2 \|z_h\|_{L^p(\Gamma(t))} \|\varphi_h\|_{L^q(\Gamma(t))}, \\ |a(z_h, \varphi_h) - a_h(Z_h, \phi_h)| &\leq ch^2 \|\nabla_\Gamma z_h\|_{L^p(\Gamma(t))} \|\nabla_\Gamma \varphi_h\|_{L^q(\Gamma(t))}, \\ |g(v_h; z_h, \varphi_h) - g_h(V_h; Z_h, \phi_h)| &\leq ch^2 \|z_h\|_{L^p(\Gamma(t))} \|\varphi_h\|_{L^q(\Gamma(t))}, \\ |b(v_h; z_h, \varphi_h) - b_h(V_h; Z_h, \phi_h)| &\leq ch^2 \|\nabla_\Gamma z_h\|_{L^p(\Gamma(t))} \|\nabla_\Gamma \varphi_h\|_{L^q(\Gamma(t))}, \end{aligned}$$

where the constant $c > 0$ is independent from $t \in [0, T]$ and the mesh width h .

Proof. These geometric estimates were established for the case $p = q = 2$ in [12, Lemma 5.5] and [21, Lemma 7.5]. To show the estimates for general p and q , the same proof apply, except the last step where we use a Hölder inequality. \square

3.4 Weighted norms and basic estimates

Similarly, as in the works of Nitsche [24], weighted Sobolev norms and their properties play a very important and central role. In this section we recall some basic results for them.

Definition 3.1 (Weight function). *For $\gamma > 0$ sufficiently big but independent of t and h we set*

$$\rho: [0, \infty) \rightarrow [0, \infty), \quad \rho^2 := \rho^2(h) := \gamma h^2 |\log h|.$$

We define a weight function $\mu = \mu(t; \cdot): \Gamma(t) \rightarrow \mathbb{R}$ via the formula

$$\mu(x) := \mu(x, y) := |x - y|^2 + \rho^2 \quad \forall x \in \Gamma(t). \quad (9) \text{ ?eq:12?}$$

The actual choice of γ is going to be clear from the proofs.

Definition 3.2 (Weighted norms, [24] Section 2.). *Let μ be a weight function and $\alpha \in \mathbb{R}$. We define the norms*

$$\begin{aligned} \|u\|_{L^2, \alpha}^2 &= \int_\Gamma \mu^{-\alpha} |u|^2, \\ \|u\|_{H^1, \alpha}^2 &= \|u\|_{L^2, \alpha}^2 + \|\nabla_\Gamma u\|_{L^2, \alpha}^2, \quad \|u\|_{H^2, \alpha}^2 = \|u\|_{H^1, \alpha}^2 + \|\nabla_\Gamma^2 u\|_{L^2, \alpha}^2. \end{aligned}$$

Lemma 3.4. *Let $\dim \Gamma(t) = 2$. Let $\phi_h \in S_h(t)$ with corresponding lift $\varphi_h \in S_h^l(t)$. Then there exist constants $c > 0$ independent of t , h and γ such that*

$$\|\varphi_h\|_{L^\infty(\Gamma(t))} \leq ch |\log h| \|\varphi_h\|_{L^2, 2}, \quad (10) \text{ ?eq:56?}$$

$$\|\varphi_h\|_{W^{1, \infty}(\Gamma(t))} \leq c\gamma |\log h|^{1/2} \|\varphi_h\|_{H^1, 1}. \quad (11) \text{ ?eq:43?}$$

Proof. There is a point $y_{0,h} \in E_0 \subset \Gamma_h(t)$ such that

$$\|\phi_h\|_{W^{1, \infty}(\Gamma_h(t))} = |\phi_h(y_{0,h})| + |\nabla_{\Gamma_h} \phi_h(y_{0,h})| = \|\phi_h\|_{W^{1, \infty}(E_0)}.$$

Note that on E_0 the estimate $\mu_h(x_h) \leq c\rho^2$ holds for $h < h_0$, h_0 sufficiently small. Then the second bound yields from using inverse inequality (Lemma 3.11) and (54). The bound (10) is proved using similar arguments. \square

Lemma 3.5. *Let $\dim \Gamma(t) = 2$. Let $u: \Gamma(t) \rightarrow \mathbb{R}$ be a sufficiently smooth function. Then the following estimates hold, with a sufficiently small $h_0 > 0$,*

$$\|u\|_{L^2,2} \leq c\rho^{-1}\|u\|_{L^\infty(\Gamma(t))}, \quad (12) \text{ ?eq:33?}$$

$$\|u\|_{H^1,1} \leq c|\log \rho|^{1/2}\|u\|_{W^{1,\infty}(\Gamma(t))}, \quad (13) \text{ ?eq:36?}$$

for $0 < h < h_0$, where the constant $c = c(h_0) > 0$ is independent of t , h and γ .

Proof. For $\alpha = 1$ or 2 we obviously have

$$\|u\|_{L^2,\alpha} \leq \|u\|_{L^\infty(\Gamma(t))}\|u\|_{L^\infty(\Gamma(t))}\|\mu^{-\alpha}\|_{L^1(\Gamma(t))}^{1/2}.$$

Then a straightforward calculation, using Appendix B shows both estimates. \square

Naturally, there is a weighted version of the Cauchy–Schwarz inequality, namely we have

$$\begin{aligned} |a^*(z_h, \varphi_h)| &\leq \|z_h\|_{H^1,\alpha}\|\varphi_h\|_{H^1,-\alpha}, \\ |a_h^*(Z_h, \phi_h)| &\leq c\|z_h\|_{H^1,\alpha}\|\varphi_h\|_{H^1,-\alpha}, \end{aligned} \quad (14) \text{ ?eq: weighted C--S estimate?}$$

and similarly for the bilinear forms g and b . Furthermore, this yields a weighted version of the geometric errors of the bilinear forms (Lemma 3.3).

Lemma 3.6. *The following estimates hold, with a constant $c > 0$ independent of t , h and γ ,*

$$|a^*(z_h^l, \phi_h^l) - a_h^*(Z_h, \phi_h)| \leq ch^2\|z_h^l\|_{H^1,\alpha}\|\phi_h^l\|_{H^1,-\alpha}, \quad (15) \text{ ?eq:61?}$$

$$|(g+b)(v_h; z_h^l, \phi_h^l) - (g_h+b_h)(V_h; Z_h, \phi_h)| \leq ch^2\|z_h^l\|_{H^1,\alpha}\|\phi_h^l\|_{H^1,-\alpha}. \quad (16) \text{ ?eq:62?}$$

Lemma 3.7. (i) *Derivatives of μ^{-1} are bounded as*

$$|\nabla_\Gamma \mu^{-1}| \leq 2\mu^{-1,5}, \quad |\Delta_\Gamma \mu^{-1}| \leq c\mu^{-2} \quad (17) \text{ ?eq:13?}$$

with $c > 0$ independent of t , h and γ .

(ii) *For arbitrary $u \in H^1(\Gamma(t))$ the following norm inequalities hold:*

$$\|\mu^{-1}u\|_{H^1,-1} \leq c(\|u\|_{L^2,2} + \|u\|_{H^1,1}), \quad (18) \text{ ?eq:44?}$$

$$\|\mu^{-2}u\|_{L^2,-1} \leq \rho^{-1}\|u\|_{L^2,2}. \quad (19) \text{ ?eq:51?}$$

Proof. (i): The first estimate follows from

$$|\nabla_\Gamma \mu^{-1}| \leq |\nabla \mu^{-1}| \leq \frac{2|x-y|}{\mu^2} \leq \frac{2\sqrt{\mu}}{\mu^2}.$$

For the second inequality consider the formula,

$$\Delta_\Gamma f = \Delta \bar{f} - \nabla^2 \bar{f}(\nu, \nu) - H\nu \cdot \nabla \bar{f},$$

where $\bar{f}: U \rightarrow \mathbb{R}$ is an extension of the sufficiently smooth function f to an open neighborhood $U \subset \mathbb{R}^{m+1}$ of $\Gamma(t)$, $\nabla^2 \bar{f}$ denotes the Hessian of \bar{f} and H denotes the trace of the Weingarten map of $\Gamma(t)$.

(ii) In order to show these estimates we use the bounds (17) obtained above. \square

3.5 Interpolation and inverse estimates

Here we collect some results involving evolving surface finite element functions.

For a sufficiently regular function $u: \Gamma(t) \rightarrow \mathbb{R}$ we denote by $\tilde{I}_h u \in S_h(t)$ its interpolation on $\Gamma_h(t)$. Then the finite element interpolation is given by $I_h u = (\tilde{I}_h u)^l \in S_h^l(t)$, having the error estimate below, cf. [11].

Lemma 3.8. *For $m \leq 3$ and $p \in \{2, \infty\}$, there exists a constant $c > 0$ independent of h and t such that for $u \in W^{2,p}(\Gamma(t))$:*

$$\begin{aligned} & \|u - I_h u\|_{L^p(\Gamma(t))} + h \|\nabla_\Gamma(u - I_h u)\|_{L^p(\Gamma(t))} \\ & \leq ch^2 \left(\|\nabla_\Gamma^2 u\|_{L^p(\Gamma(t))} + h \|\nabla_\Gamma u\|_{L^p(\Gamma(t))} \right). \end{aligned}$$

The interpolation estimates hold also if weighted norms are considered.

Lemma 3.9. *There exists a constant $c > 0$ such that for $u \in W^{2,\infty}(\Gamma(t))$ it holds*

$$\|u - I_h u\|_{L^2,2}^2 + \|u - I_h u\|_{H^1,1}^2 \leq ch^2 |\log h| \|u\|_{W^{2,\infty}(\Gamma(t))}^2. \quad (20) \text{ ?eq:57?}$$

Proof. Use a Hölder inequality, Lemma 3.8 and Lemma 3.5 (12), (13) with the choice $u \equiv 1$. \square

Lemma 3.10. *There exists $h_0 > 0$, $\gamma_0 > 0$ such that for all $\alpha \in \mathbb{R}$ there exists a constant $c = c(h_0, \gamma_0) > 0$ independent of t and h such that for all $\gamma > \gamma_0$ for the weight μ , c.f. (9), and for all $h < h_0$ the following inequalities holds:*

(i) *Let $u \in H^1(\Gamma(t))$ be curved element-wise H^2 . The interpolation $I_h u \in S_h^l(t)$ satisfies*

$$\|u - I_h u\|_{L^2,\alpha} + h \|\nabla_\Gamma(u - I_h u)\|_{L^2,\alpha} \leq ch^2 (\|\nabla_\Gamma^2 u\|_{L^2,\alpha} + ch \|\nabla_\Gamma u\|_{L^2,\alpha}), \quad (21) \text{ ?eq:17?}$$

where $\|\nabla_\Gamma^2 u\|_{L^2,\alpha}$ is understood curved element-wise.

(ii) *For any $\varphi_h \in S_h^l(t)$ the following estimate holds:*

$$\|\mu^{-1} \varphi_h - I_h(\mu^{-1} \varphi_h)\|_{H^1,-1} \leq c \left(\frac{h}{\rho} + h \right) (\|\varphi_h\|_{L^2,2} + \|\nabla_\Gamma \varphi_h\|_{L^2,1}). \quad (22) \text{ ?eq:18?}$$

Proof. (i): To prove inequality (21) it suffices to show that there exists a constant $c = c(\alpha) > 0$ independent of t, h such that for each element $K \in \mathcal{T}_h(t)$ it holds

$$\begin{aligned} & \int_{K^l} \mu^\alpha ((w - I_h w)^2 + h |\nabla_\Gamma(w - I_h w)|^2) \\ & \leq ch^2 \int_{K^l} \mu^\alpha (|\nabla_\Gamma^2 w|^2 + ch |\nabla_\Gamma w|^2), \end{aligned}$$

where $K^l \subset \Gamma(t)$ denote the lifted curved element of K . It is easy to show that there exists $\gamma_0 = \gamma_0(h_0) > 0$ and $c = c(\gamma_0) > 0$ such that for all $\gamma > \gamma_0$ it holds

$$\max_{K \in \mathcal{T}_h} \left(\frac{\max_{x \in K^l} \mu(x, y)}{\min_{x \in K^l} \mu(x, y)} \right) \leq c.$$

A straightforward calculation finishes the proof.

(ii): For an arbitrary function $f: \Gamma_h(t) \rightarrow \mathbb{R}$, which is element-wise H^2 , a short calculation, similar to the one done in Dziuk [8, Lemma 3], shows that

$$|(\nabla_\Gamma)_i(\nabla_\Gamma)_j(f^l)| \leq c(|((\nabla_{\Gamma_h})_i(\nabla_{\Gamma_h})_j f)^l| + ch|\nabla_\Gamma(f^l)|),$$

for a sufficiently small $h_0 > h > 0$. A straightforward calculation combined with (i) and (17) shows the claim. \square

The following general version of inverse estimates for finite element functions plays a key role later on, cf. [25].

Lemma 3.11 (Inverse estimate). *There exists $c > 0$ such that for each triangle $E_h(t) \subset \Gamma_h(t)$ the following inequality holds*

$$\|\varphi_h(t)\|_{W^{k,p}(E_h(t))} \leq ch^{m-k-2(1/q-1/p)} \|\varphi_h(t)\|_{W^{m,q}(E_h(t))} \quad (\forall \varphi_h \in S_h(t)).$$

Lemma 3.12. *There exists $c > 0$ with*

$$\|\varphi_h\|_{L^\infty(\Gamma(t))} \leq \left| \frac{1}{V} \int_{\Gamma(t)} \varphi_h(y) dV(y) \right| + c|\log h|^{1/2} \|\nabla_\Gamma \varphi_h\|_{L^\infty(\Gamma(t))}.$$

Proof. Follow the steps in Schatz, Thomée, Wahlbin [25] using the Green's function from Theorem A.1 and calculating with geodesic polar coordinates. \square

3.6 Estimates for an L^2 -projection

This section shows some technical results for the L^2 -projection, which is denoted by P_0 (in contrast with the Ritz map which will be denoted by P_1).

Definition 3.3 (L^2 -projection). *We define $P_0(t): L^2(\Gamma_h(t)) \rightarrow S_h(t)$ as follows: Let $u_h \in L^2(\Gamma_h(t))$ be given. Then there exists a unique finite element function $P_0(t)u \in S_h(t)$ such that for all $\phi_h \in S_h(t)$ it holds*

$$m_h(P_0(t)u_h, \phi_h) = m_h(u_h, \phi_h). \quad (23) \text{ ?eq:47?}$$

The following important L^p -stability bound and exponential decay property from Douglas, Dupont and Wahlbin [7, equation (6) and (7)] holds without any serious modification.

Theorem 3.1. *For $p \in [1, \infty]$ let $u_h \in L^p(\Gamma_h(t))$. Then there exists a constant $c > 0$ independent of h and t such that*

$$\|P_0(t)u_h\|_{L^p(\Gamma_h(t))} \leq c\|u_h\|_{L^p(\Gamma_h(t))}.$$

Further there exists $c_2, c_3 > 0$ independent of h and t such that for $A_h^1(t)$ and $A_h^2(t)$ disjoint subsets of $\Gamma_h(t)$ with $\text{supp}(u_h) \subseteq A_h^1$ we have

$$\|P_0(t)u_h\|_{L^2(A_h^2(t))} \leq c_2 e^{-c_3 \text{dist}_h(A_h^1, A_h^2)h^{-1}} \|u_h\|_{L^2(A_h^1(t))}, \quad (24) \text{ ?eq:14?}$$

where $\text{dist}_h(x, y) = \text{dist}_{\Gamma_h(t)}(x, y)$ is the intrinsic Riemannian distance of $\Gamma_h(t)$.

For the proof of our discrete weak maximum principle we are going to use a different weight function then (9). Let $[0, T] \rightarrow \mathbb{R}^{m+1}, t \mapsto y(t)$ be a curve with the property $y(t) \in \Gamma(t)$. In the following we write y instead of $y(t)$. We define

$$\sigma(x) := \sigma^y(x) := \sigma(x, y) := (|x - y|^2 + h^2)^{1/2}. \quad (25) \text{ ?eq:10?}$$

We gather some estimates concerning σ in the next lemma.

Lemma 3.13. *There exists a constant $c > 0$ independent of t and h such that the following estimates hold*

$$\|\partial^\bullet \sigma\|_{L^\infty(\Gamma(t))} \leq c, \quad \|\partial_h^\bullet \sigma\|_{L^\infty(\Gamma(t))} \leq c, \quad (26) \text{ ?eq:67?}$$

$$\|\nabla_\Gamma \sigma\|_{L^\infty(\Gamma(t))} \leq 1, \quad |\nabla_\Gamma^2 \sigma| \leq c \left(\frac{1}{\sigma} + 1 \right), \quad \|\nabla_\Gamma^2(\sigma^2)\|_{L^\infty(\Gamma(t))} \leq c. \quad (27) \text{ ?eq:68?}$$

The proof of this lemma is a straightforward calculation and is omitted here.

Lemma 3.14. *There exists $c > 0$ such for fixed $t \in [0, T]$, $x_h \in \Gamma_h(t)$, $\sigma = \sigma^{x_h}$, $\phi_h \in S_h(t)$ and $\psi_h := P_0(\rho^2 \phi_h)$ the following inequality holds:*

$$\begin{aligned} \|\rho^2 \phi_h - \psi_h\|_{L^2(\Gamma_h(t))} + h \|\nabla_{\Gamma_h}(\rho^2 \phi_h - \psi_h)\|_{L^2(\Gamma_h(t))} \\ \leq ch^2 (\|\phi_h\|_{L^2(\Gamma_h(t))} + \|\rho \nabla_{\Gamma_h} \phi_h\|_{L^2(\Gamma_h(t))}). \end{aligned}$$

Proof. Consider a triangle $E_h \subset \Gamma_h(t)$ and set $g_h := \tilde{I}_h(\rho^2 \phi_h)$. Use Lemma 3.13 and (55) and follow the steps in Schatz, Thomée and Wahlbin [25, Lemma 1.4]. \square

4 A Ritz map and some error estimates

Just as in the usual L^2 -theory the Ritz map plays a very important role for our L^∞ -error estimates. This section is devoted to the careful L^∞ - and weighted norm analysis of the errors in the Ritz map.

Definition 4.1 (Ritz map, [21]). *We define $P_{h,1}(t): H^1(\Gamma(t)) \rightarrow S_h(t)$ as follows: Let $u \in H^1(\Gamma(t))$ be given. Then there exists a unique finite element function $P_{h,1}(t)u \in S_h(t)$ such that for all $\phi_h \in S_h(t)$ with $\varphi_h = \phi_h^l$ it holds*

$$a_h^*(P_{h,1}(t)u, \phi_h) = a^*(u, \varphi_h). \quad (28) \text{ ?eq:39?}$$

This naturally defines the Ritz map on the continuous surface:

$$P_1(t)u = (P_{h,1}(t)u)^l \in S_h^l(t).$$

Note that the Ritz map does not satisfy the Galerkin orthogonality, however it satisfies, using (15), the following estimate, cf. [21]. For all $\varphi_h \in S_h^l(t)$ we have

$$|a^*(u - P_1(t)u, \varphi_h)| \leq ch^2 \|P_1(t)u\|_{H^1, \alpha} \|\varphi_h\|_{H^1, -\alpha}. \quad (29) \text{ ?eq:49?}$$

In this section we aim to bound the following errors of the Ritz map:

$$u - P_1(t)u \quad \text{and} \quad \partial_h^\bullet(u - P_1(t)u),$$

in the L^∞ - and $W^{1,\infty}$ -norms. Previously, H^1 - and L^2 -error estimates have been shown in [9, 12].

4.1 Weighted a priori estimates

Before turning to the maximum norm error estimates, we state and prove some technical regularity results involving weighted norms.

Lemma 4.1 (Weighted a priori estimates). *For $f \in L^2(\Gamma(t))$, the problem*

$$-\Delta_\Gamma w + w = f \quad \text{on } \Gamma(t),$$

has a unique weak solution $w \in H^1(\Gamma(t))$. Furthermore, $w \in H^2(\Gamma(t))$ and we have the following weighted a priori estimates

$$\|w\|_{H^1, -1} \leq c(\|f\|_{L^2, -1} + \|w\|_{L^2}) \quad (30) \text{ ?eq:26?}$$

$$\|w\|_{H^2, -1} \leq c(\|f\|_{L^2, -1} + \|w\|_{H^1}), \quad (31) \text{ ?eq:24?}$$

where the constant $c > 0$ is independent of t, h and γ .

Proof. Existence and uniqueness of a weak solution follows from [1]. Using integration by parts, Young inequality and $|\nabla_\Gamma \mu| \leq \sqrt{\mu}$ a short calculation shows (30). For the details on elliptic regularity and a derivation of the a priori estimate

$$\|w\|_{H^2} \leq c\|-\Delta_\Gamma w + w\|_{L^2},$$

where $c > 0$ is independent of t , we refer to [19, Appendix A].

Because of (30) it suffices to prove (31) for $\|\nabla_\Gamma^2 w\|_{L^2, -1}^2$ as the left-hand side instead of $\|w\|_{H^2, -1}^2$. Apply the usual elliptic a priori estimate on $(x^i - y^i)w$ for $i = 1, \dots, m+1$ to get the desired estimate. \square

Lemma 4.2. *For $g \in L^2(\Gamma(t))$ the problem*

$$-\Delta_\Gamma w + w = \mu^{-2}g.$$

has a unique weak solution $w \in H^1(\Gamma(t))$. Furthermore, $w \in H^2(\Gamma(t))$, and there exists a constant $c > 0$ independent of t and h such that

$$\|w\|_{H^1}^2 \leq c\rho^{-2}|\log \rho|\|g\|_{L^2, 2}^2. \quad (32) \text{ ?eq:45?}$$

Proof. Lemma 4.1 gives us existence, uniqueness and regularity of w . Consider the number

$$\lambda^{-1}(t) := \sup\{\|f\|_{H^1}^2 \mid f \in H^2(\Gamma(t)), \ \|-\Delta_\Gamma f + f\|_{L^2, -2}^2 \leq 1\}.$$

Inequality (32) is proven if we show

$$\lambda^{-1}(t) \leq c\rho^{-2}|\log \rho|,$$

where c is t independent. A short calculation shows that the smallest eigenvalue $\tilde{\lambda}_{\min}(t)$ of the elliptic eigenvalue problem

$$-\Delta_\Gamma f + f = \tilde{\lambda}\mu^{-2}f \quad \text{on } \Gamma(t)$$

is equal to $\lambda(t)$. The weighted Rayleigh quotient implies

$$\tilde{\lambda}_{\min} = \inf_{f \in H^1} \frac{\|f\|_{H^1}^2}{\|f\|_{L^2, 2}^2}.$$

Hence it suffices to prove

$$\|f\|_{L^2,2}^2 \leq c\rho^{-2}|\log(\rho)|\|f\|_{H^1}^2, \quad (33) \text{ ?eq:50?}$$

for a $f \in H^1$. With a Hölder estimate we arrive at

$$\|f\|_{L^2,2}^2 \leq \left(\int_{\Gamma(t)} \mu^{-2p} \right)^{1/p} \left(\int_{\Gamma(t)} f^{2q} \right)^{1/q} = \left(\int_{\Gamma(t)} \mu^{-2p} \right)^{1/p} \|f\|_{L^{2q}(\Gamma(t))}^2,$$

where $1 < p, q < \infty$ satisfies $p^{-1} + q^{-1} = 1$. We take the choice $q = \sqrt{|\log \rho|}$. It is easy to prove the following quantitative Sobolev-Nierenberg inequality for moving surfaces:

$$\|f\|_{L^q(\Gamma(t))} \leq cq\|f\|_{H^1(\Gamma(t))},$$

where c is independent of t and q . A straightforward calculation with geodesic polar coordinates using Lemma B.2 and Lemma B.1 shows inequality (33). \square

4.2 Maximum norm error estimates

Before showing L^∞ - and $W^{1,\infty}$ -norm error estimates for the Ritz map, we show similar estimates for weighted norms. Then, by connecting the norms, use these results to obtain our original goal.

Throughout this subsection, we write $P_1 u$ instead of $P_1(t)u$.

Lemma 4.3. *There exists $h_0 > 0$ sufficiently small and $\gamma_0 > 0$ sufficiently large and a constant $c = c(h_0, \gamma_0) > 0$ such that for $u \in W^{2,\infty}(\Gamma(t))$ it holds*

$$\|u - P_1 u\|_{L^2,2}^2 + \|u - P_1 u\|_{H^1,1}^2 \leq ch^2 |\log h| \|u\|_{W^{2,\infty}(\Gamma(t))}^2. \quad (34) \text{ ?eq:53?}$$

Proof. Step 1: Our goal is to show

$$\|u - P_1 u\|_{H^1,1}^2 \leq ch^2 |\log h| \|u\|_{W^{2,\infty}(\Gamma(t))}^2 + \hat{c} \|u - P_1 u\|_{L^2,2}^2. \quad (35) \text{ ?eq:46?}$$

Similarly as in Nitsche [24, Theorem 1], (17) and partial integration yields

$$\begin{aligned} \|u - P_1 u\|_{H^1,1}^2 &\leq a^*(u - P_1 u, \mu^{-1}(u - P_1 u)) + c \int_{\Gamma(t)} (\Delta_\Gamma \mu^{-2})(u - P_1 u)^2 \\ &\leq a^*(u - P_1 u, \mu^{-1}(u - P_1 u)) + c \|u - P_1 u\|_{L^2,2}^2. \end{aligned}$$

For simplicity we set $e = u - P_1 u$, and use $I_h u = (\tilde{I}_h u)^l$ to obtain

$$\begin{aligned} a^*(e, \mu^{-1}e) &= a^*(e, \mu^{-1}(u - I_h u)) \\ &\quad + a^*\left(e, \mu^{-1}(I_h u - P_1 u) - I_h(\mu^{-1}(I_h u - P_1 u))\right) \\ &\quad + a^*\left(e, I_h(\mu^{-1}(I_h u - P_1 u))\right) = I_1 + I_2 + I_3. \end{aligned}$$

Using Lemma 3.6 (14), Lemma 3.7 (18), Lemma 3.9 (20) and ε -Young inequality we estimate as

$$|I_1| \leq \varepsilon \|e\|_{H^1,1}^2 + ch^2 |\log h| \|u\|_{W^{2,\infty}(\Gamma(t))}^2.$$

For the second term use in addition Lemma 3.10 (22) and a $0 < h < h_0$ sufficiently small to get

$$|I_2| \leq \varepsilon \|e\|_{H^1,1}^2 + c(h^2 |\log h| \|u\|_{W^{2,\infty}(\Gamma(t))} + \|e\|_{L^2,2}).$$

For the last term use in addition Lemma 3.6 (29) to reach at

$$|I_3| \leq \varepsilon \|e\|_{H^1,1}^2 + c(h^2 |\log h| \|u\|_{W^{2,\infty}(\Gamma(t))} + \|e\|_{L^2,2})$$

These estimates together, and absorbing $\|e\|_{H^1,1}^2$, imply (35).

Step 2: Using an Aubin–Nitsche argument we prove that there exists $\gamma > \gamma_0 > 0$ sufficiently large such that for all $\delta > 0$ the following estimate holds

$$\|u - P_1 u\|_{L^2,2}^2 \leq ch^4 \|u\|_{W^{2,\infty}(\Gamma(t))}^2 + \delta \|u - P_1 u\|_{H^1,1}^2. \quad (36) \text{ ?eq:52?}$$

Let $w \in H^2(\Gamma(t))$ be the weak solution of

$$-\Delta_\Gamma w + w = \mu^{-2} e.$$

Then by testing with e we obtain

$$\|e\|_{L^2,2}^2 = (a^*(e, w) - a^*(e, I_h w)) + a^*(e, I_h w) = a^*(e, w - I_h w) + a^*(e, I_h w)$$

In addition to the already mentioned lemmata in Step 1 use Lemma 4.1 (31), Lemma 3.7 (19), Lemma 4.2 (32) and a sufficiently large $\gamma > \gamma_0 > 0$ to estimate

$$|a^*(e, w - I_h w)| \leq \frac{1}{4} \|e\|_{L^2,2}^2 + \frac{\delta}{2} \|e\|_{H^1,1}^2.$$

For the other term we estimate

$$|a^*(e, I_h w)| \leq ch^2 \|e\|_{H^1} \|I_h w\|_{H^1} \leq ch^4 \|u\|_{W^{2,\infty}(\Gamma(t))}^2 + \frac{1}{4} \|e\|_{L^2,2}^2.$$

By absorption, this implies (36).

The final estimate is shown by combining (35) and (36), and choosing $\delta > 0$ such that $\hat{c}\delta < 1$. Then an absorbtion finishes the proof. \square

Theorem 4.1. *There exist constants $c > 0$ independent of h and t such that*

$$\begin{aligned} \|u - (P_{h,1}(t)u)^l\|_{L^\infty(\Gamma(t))} &\leq ch^2 |\log h|^{3/2} \|u\|_{W^{2,\infty}(\Gamma(t))}, \quad (u \in W^{2,\infty}(\Gamma(t))). \\ \|u - (P_{h,1}(t)u)^l\|_{W^{1,\infty}(\Gamma(t))} &\leq ch |\log h| \|u\|_{W^{2,\infty}(\Gamma(t))}, \end{aligned}$$

Proof. Using Lemma 3.8, Lemma 3.4 (11) and Lemma 3.5 (13) we get

$$\begin{aligned} \|u - P_1 u\|_{W^{1,\infty}(\Gamma(t))} &\leq \|u - I_h u\|_{W^{1,\infty}(\Gamma(t))} + c \|\tilde{I}_h u - P_{h,1} u\|_{W^{1,\infty}(\Gamma_h(t))} \\ &\leq ch \|u\|_{W^{2,\infty}(\Gamma(t))} + c |\log h|^{1/2} \|\tilde{I}_h u - P_{h,1} u\|_{H^1,1} \\ &\leq ch |\log h| \|u\|_{W^{2,\infty}(\Gamma_h(t))} + c \|u - (P_{h,1} u)^l\|_{H^1,1}. \end{aligned}$$

For the $W^{1,\infty}$ -estimate use Lemma 4.3 to estimate the weighted norms. The L^∞ -estimate is obtained in a similar way. \square

4.3 Maximum norm material derivative error estimates

Since in general time derivatives are not commuting with the Ritz map, i.e. $\partial_h^\bullet P_1 u \neq P_1 \partial_h^\bullet u$, we bound the error $\partial_h^\bullet(u - P_1 u)$. Again we first show our estimates in the weighted norms, and then use these results for the L^∞ - and $W^{1,\infty}$ -norm error estimates.

For this subsection we write $P_{h,1}u$ instead of $P_{h,1}(t)u$ and further $P_1 u$ instead of $P_1(t)u$.

We first state a substitute for our weighted pseudo Galerkin inequality (29).

Lemma 4.4. *There exists a constant $c > 0$ independent of h and t such that for all $u \in W^{2,\infty}(\mathcal{G}_T)$ and $\varphi_h \in S_h^l(t)$ it holds*

$$\begin{aligned} |a^*(\partial_h^\bullet(u - P_1 u), \varphi_h)| &\leq c \left(h^2 \|\partial_h^\bullet(u - P_1 u)\|_{H^1,1} \right. \\ &\quad \left. + h |\log h|^{1/2} (\|u\|_{W^{2,\infty}(\Gamma(t))} + \|\partial^\bullet u\|_{W^{1,\infty}(\Gamma(t))}) \right) \|\varphi_h\|_{H^1,-1}. \end{aligned} \quad (37) \text{ ?eq:59?}$$

Proof. The main idea is given by Dziuk and Elliott in [12]. Using (5) and Lemma 3.5 (13) it is easy to verify

$$\begin{aligned} \|\partial_h^\bullet P_1 u\|_{H^1,1} &\leq \|\partial_h^\bullet u - \partial_h^\bullet P_1 u\|_{H^1,1} \\ &\quad + c |\log h|^{1/2} (\|\partial^\bullet u\|_{W^{1,\infty}(\Gamma(t))} + h \|u\|_{W^{2,\infty}(\Gamma(t))}). \end{aligned} \quad (38) \text{ ?eq:63?}$$

Let $\phi_h \in S_h(t)$ such that $\varphi_h = \phi_h^l$. Taking time derivative of the definition of the Ritz map (28), using the discrete transport properties (7) Lemma 3.2, and the definition of the Ritz map, we obtain

$$\begin{aligned} a^*(\partial_h^\bullet u - \partial_h^\bullet P_1 u, \varphi_h) &= a_h^*(\partial_h^\bullet P_{h,1} u, \phi_h) - a^*(\partial_h^\bullet P_1 u, \varphi_h) \\ &\quad + (g_h + b_h)(V_h; u^{-l}, \phi_h) - (g + b)(v_h; u, \varphi_h) \\ &\quad - (g_h + b_h)(V_h; u^{-l} - P_{h,1} u, \phi_h). \end{aligned} \quad (39) \text{ ?eq:65?}$$

Then estimate using Lemma 3.6 (15), (16), Lemma 4.3 (34) and the above inequality to finish the proof (cf. [22, Theorem 7.2]). \square

Lemma 4.5. *For $k \in \{0, 1\}$ there exists $c = c(k) > 0$ independent of t and h such that for $u \in W^{3,\infty}(\mathcal{G}_T)$ the following inequalities hold*

$$\|\partial_h^\bullet u - I_h \partial^\bullet u\|_{W^{k,\infty}(\Gamma(t))} \leq ch^{2-k} (\|u\|_{W^{2,\infty}(\Gamma(t))} + \|\partial^\bullet u\|_{W^{2,\infty}(\Gamma(t))}), \quad (40) \text{ ?eq:35?}$$

$$\begin{aligned} \|\partial_h^\bullet u - I_h \partial^\bullet u\|_{L^2,2}^2 + \|\partial_h^\bullet u - I_h \partial^\bullet u\|_{H^1,1}^2 \\ \leq ch^2 |\log h| (\|u\|_{W^{2,\infty}(\Gamma(t))} + \|\partial^\bullet u\|_{W^{2,\infty}(\Gamma(t))}). \end{aligned} \quad (41) \text{ ?eq:58?}$$

Proof. Using (5) we get

$$\begin{aligned} \|\partial_h^\bullet u - I_h \partial^\bullet u\|_{W^{k,\infty}(\Gamma(t))} \\ \leq \|(v - v_h) \cdot \nabla \Gamma u\|_{W^{k,\infty}(\Gamma(t))} + \|\partial^\bullet u - I_h \partial^\bullet u\|_{W^{k,\infty}(\Gamma(t))}. \end{aligned}$$

Use Lemma 3.8 and (6) to show the first estimate.

For the second inequality use a Hölder estimate, and (40) with Lemma 3.5 (12) and (13). \square

Lemma 4.6. *There exists $h_0 > 0$ sufficiently small and $\gamma_0 > 0$ sufficiently large and a constant $c = c(h_0, \gamma_0) > 0$ such that for $u \in W^{3,\infty}(\mathcal{G}_T)$ the following holds*

$$\begin{aligned} \|\partial_h^\bullet u - \partial_h^\bullet P_1 u\|_{L^2,2}^2 + \|\partial_h^\bullet u - \partial_h^\bullet P_1 u\|_{H^1,1}^2 \\ \leq ch^2 |\log h|^4 (\|u\|_{W^{2,\infty}(\Gamma(t))}^2 + \|\partial^\bullet u\|_{W^{2,\infty}(\Gamma(t))}^2). \end{aligned} \quad (42) \text{ ?eq:55?}$$

Proof. This proof has a similar structure as Lemma 4.3, and since it also uses similar arguments, we only give references if new lemmata are needed. For the ease of presentation we set $e = u - P_1 u$ and split the error as follows

$$\partial_h^\bullet e = (\partial_h^\bullet u - I_h \partial^\bullet u) + (I_h \partial^\bullet u - \partial_h^\bullet P_1 u) =: \sigma + \theta_h.$$

Step 1: Our goal is to prove

$$\|\partial_h^\bullet e\|_{H^1,1}^2 \leq ch^2 |\log h| (\|u\|_{W^{2,\infty}(\Gamma(t))}^2 + \|\partial^\bullet u\|_{W^{2,\infty}(\Gamma(t))}^2) + \hat{c} \|\partial_h^\bullet e\|_{L^2,2}^2. \quad (43) \text{ ?eq:32?}$$

We start with

$$\|\partial_h^\bullet e\|_{H^1,1}^2 \leq a^* \left(\partial_h^\bullet e, \mu^{-1} \partial_h^\bullet e \right) + c \|\partial_h^\bullet e\|_{L^2,2}^2$$

and continue with

$$\begin{aligned} a^* (\partial_h^\bullet e, \mu^{-1} \partial_h^\bullet e) &= a^* (\partial_h^\bullet e, \mu^{-1} \sigma) \\ &\quad + a^* (\partial_h^\bullet e, \mu^{-1} \theta_h - I(\mu^{-1} \theta_h)) \\ &\quad + a^* (\partial_h^\bullet e, I(\mu^{-1} \theta_h)) = I_1 + I_2 + I_3. \end{aligned}$$

We estimate the three terms separately. For the first ε -Young inequality and Lemma 4.5 (41) yields

$$|I_1| \leq \varepsilon \|\partial_h^\bullet e\|_{H^1,1}^2 + ch^2 |\log h| (\|u\|_{W^{2,\infty}(\Gamma(t))}^2 + \|\partial^\bullet u\|_{W^{2,\infty}(\Gamma(t))}^2).$$

For a sufficiently small $0 < h < h_0$ we obtain

$$|I_2| \leq \varepsilon \|\partial_h^\bullet e\|_{H^1,1}^2 + c (\|\partial_h^\bullet e\|_{L^2,2}^2 + h^2 |\log h| (\|u\|_{W^{2,\infty}(\Gamma(t))}^2 + \|\partial^\bullet u\|_{W^{2,\infty}(\Gamma(t))}^2)).$$

Using Lemma 4.4 (37) and a $0 < h < h_1$ sufficiently small we arrive at

$$|I_3| \leq \varepsilon \|\partial_h^\bullet e\|_{H^1,1}^2 + c (\|\partial_h^\bullet e\|_{L^2,2}^2 + h^2 |\log h| (\|u\|_{W^{2,\infty}(\Gamma(t))}^2 + \|\partial^\bullet u\|_{W^{2,\infty}(\Gamma(t))}^2)).$$

These estimates together, and absorbing $\|\partial_h^\bullet e\|_{H^1,1}$, imply (43).

Step 2: Using again an Aubin–Nitsche like argument we show that, for any $\delta > 0$ sufficiently small, we have

$$\|\partial_h^\bullet e\|_{L^2,2}^2 \leq \delta \|\partial_h^\bullet e\|_{H^1,1}^2 + ch^2 |\log h|^4 (\|u\|_{W^{2,\infty}(\Gamma(t))}^2 + \|\partial^\bullet u\|_{W^{2,\infty}(\Gamma(t))}^2). \quad (44) \text{ ?eq:64?}$$

Let $w \in H^2(\Gamma(t))$ be the weak solution of

$$-\Delta_\Gamma w + w = \mu^{-2} \partial_h^\bullet e.$$

Then we have

$$\|\partial_h^\bullet e\|_{L^2,2}^2 = a^* (\partial_h^\bullet e, w - I_h w) + a^* (\partial_h^\bullet e, I_h w).$$

Again let $\varepsilon > 0$ be a small number. For $\gamma > \gamma_0$ sufficiently big we get

$$|a^*(\partial_h^\bullet e, w - I_h w)| \leq \varepsilon \|\partial_h^\bullet e\|_{H^1,1}^2 + \delta \|\partial_h^\bullet e\|_{L^2,2}^2$$

Using equation (39) and proceeding similar like in Dziuk and Elliott [12, Theorem 6.2], by adding and subtracting terms, we get

$$\begin{aligned} a^*(\partial_h^\bullet e, I_h w) &= - \left(a^*(\partial_h^\bullet P_1 u, I_h w) - a_h^*(\partial_h^\bullet P_{h,1} u, \tilde{I}_h w) \right. \\ &\quad + (g+b)(v_h; u, I_h w) - (g_h + b_h)(V_h; u^{-l}, \tilde{I}_h w) \\ &\quad + (g_h + b_h)(V_h; u^{-l} - P_{h,1} u, \tilde{I}_h w) - (g+b)(v_h; u - P_1 u, I_h w) \\ &\quad + (g+b)(v_h; u - P_1 u, I_h w) - (g+b)(v; u - P_1 u, I_h w) \\ &\quad + (g+b)(v; u - P_1 u, I_h w) - (g+b)(v; u - P_1 u, w) \\ &\quad \left. + (g+b)(v; u - P_1 u, w) \right) \\ &= J_1 + J_2 + J_3 + J_4 + J_5 + J_6. \end{aligned}$$

Use Lemma 3.6 (16), (38), Lemma 4.3 (34) and the inequality

$$h \|I_h w\|_{H^1,1} \leq \varepsilon \|\partial_h^\bullet e\|_{L^2,2},$$

for $\gamma > \gamma_1$ sufficiently big, we reach at

$$|J_1| + \dots + |J_4| \leq \delta \|\partial_h^\bullet e\|_{H^1,1}^2 + \varepsilon \|\partial_h^\bullet e\|_{L^2,2}^2 + ch^2(\|u\|_{W^{2,\infty}}^2 + \|\partial^\bullet u\|_{W^{1,\infty}}^2).$$

With the same arguments like for $a^*(\partial_h^\bullet e, w - I_h w)$ we estimate

$$|J_5| \leq \varepsilon \|\partial_h^\bullet e\|_{H^1,1}^2 + \delta \|\partial_h^\bullet e\|_{L^2,2}^2,$$

for $\gamma > \gamma_2$ sufficiently big. For $\gamma > \gamma_3$ sufficiently big we estimate the last term as follows

$$\begin{aligned} |J_6| &\leq c \|e\|_{L^2,1} \|w\|_{H^2,-1} \\ &\leq c \|e\|_{L^\infty} |\log \rho|^{1/2} \|w\|_{H^2,-1} \\ &\leq ch^2 |\log h|^{3/2} \|u\|_{W^{2,\infty}} \|w\|_{H^2,-1} \\ &\leq \varepsilon \|\partial_h^\bullet e\|_{L^2,2}^2 + ch^2 |\log h|^4 \|u\|_{W^{2,\infty}}^2. \end{aligned}$$

By absorption, these estimates together imply (44).

The final estimate is shown by combining (43) and (44), and choosing $\delta > 0$ such that $\hat{c}\delta < 1$. Then an absorbtion finishes the proof. \square

From the weighted version of the error estimate in the material derivatives, the L^∞ -norm estimate follows easily.

Theorem 4.2 (Errors in the material derivative of the Ritz projection). *Let $z \in W^{3,\infty}(\mathcal{G}_T)$. For a sufficiently small $h < h_0$ and a sufficiently big $\gamma > \gamma_0$ there exists $c = c(h_0, \gamma_0) > 0$ independent of t and h such that*

$$\begin{aligned} \|\partial_h^\bullet(z - (P_{h,1}(t)z)^l)\|_{L^\infty(\Gamma(t))} &\leq ch^2 |\log h|^3 (\|z\|_{W^{2,\infty}(\Gamma(t))} + \|\partial^\bullet z\|_{W^{2,\infty}(\Gamma(t))}), \\ \|\partial_h^\bullet(z - (P_{h,1}(t)z)^l)\|_{W^{1,\infty}(\Gamma(t))} &\leq ch |\log h|^{5/2} (\|z\|_{W^{2,\infty}(\Gamma(t))} + \|\partial^\bullet z\|_{W^{2,\infty}(\Gamma(t))}). \end{aligned}$$

Proof. The above results are shown by exactly following the proof of Theorem 4.1, Lemma 4.6 (42) being the main tool. \square

5 Maximum norm parabolic stability

The purpose of this section is to derive a ESFEM weak discrete maximum principle. The proof is modeled on the weak discrete maximum principle from Schatz, Thomée, Wahlbin [25]. For this we are going to need a well known matrix formulation of (3), which is due to Dziuk and Elliott [9]. It was first used in Dziuk, Lubich, Mansour [13] for theoretical reasons, namely a time discretization of (3). Using the matrix formulation we derive a discrete adjoint problem of (3), which does not arise in Schatz, Thomée, Wahlbin [25], but arises here, since the ESFEM evolution operator is not self adjoint. Then we deduce a corresponding a priori estimate and finally prove our weak discrete maximum principle.

5.1 A discrete adjoint problem

A matrix ODE version of (3) can be derived by setting

$$U_h(\cdot, t) = \sum_{j=1}^N \alpha_j(t) \chi_j(\cdot, t),$$

testing with the basis function $\phi_h = \chi_j$, where $S_h(t) = \text{lin}\{\chi_j \mid j = 1, \dots, N\}$, and using the transport property (2).

Proposition 5.1 (ODE system). *The spatially semidiscrete problem (3) is equivalent to the following linear ODE system for the vector $\alpha(t) = (\alpha_j(t)) \in \mathbb{R}^N$, collecting the nodal values of $U_h(\cdot, t)$:*

$$\begin{cases} \frac{d}{dt} (M(t)\alpha(t)) + A(t)\alpha(t) = 0 \\ \alpha(0) = \alpha_0 \end{cases} \quad (45) \text{ ?eq: ODE system?}$$

where the evolving mass matrix $M(t)$ and stiffness matrix $A(t)$ are defined as

$$M(t)_{kj} = \int_{\Gamma_h(t)} \chi_j \chi_k, \quad A(t)_{kj} = \int_{\Gamma_h(t)} \nabla_{\Gamma_h} \chi_j \cdot \nabla_{\Gamma_h} \chi_k.$$

Definition 5.1. Let $0 \leq s \leq t \leq T$. For given initial value $w_h \in S_h(s)$ at time s , there exists unique¹ solution u_h . This defines a linear evolution operator

$$E_h(t, s): S_h(s) \rightarrow S_h(t), \quad w_h \mapsto u_h(t).$$

We define the adjoint of $E_h(t, s)$

$$E_h(t, s)^*: S_h(t) \rightarrow S_h(s)$$

via the equation

$$m_h(t; E_h(t, s)\varphi_h(s), w_h(t)) = m_h(s; \varphi_h(s), E_h(t, s)^* w_h(t)), \quad (46) \text{ ?eq:21?}$$

where $\varphi_h(s) \in S_h(s)$ and $w_h(t) \in S_h(t)$ are some arbitrary finite element functions.

¹cf. Dziuk and Elliott [9].

Lemma 5.1 (Adjoint problem). *Let $s \in [0, t]$ where $t \in [0, T]$ and $w_h(t) \in S_h(t)$. Then $s \mapsto E(t, s)^* w_h(t)$ is the unique solution of*

$$\begin{cases} m_h(s; \partial_h^{\bullet, s} u_h, \varphi_h) - a_h(s; u_h, \varphi_h) = 0, & \text{on } \Gamma(s) \\ u_h(t) = w_h(t), & \text{on } \Gamma(t). \end{cases} \quad (47) \text{ ?eq:22?}$$

where $\partial_h^{\bullet, s}$ is the discrete material derivative with respect to s .

Remark 5.1. *The problem (47) has the structure of a backward heat equation, where s is going backward in time. Hence we considered (47) as a PDE of parabolic type. We recall, that using Lemma 3.2 we may write equation (3) equivalently as*

$$\begin{cases} m_h(t; \partial_h^{\bullet} u_h + (\nabla_{\Gamma_h} \cdot V_h) u_h, \varphi_h) + a_h(t; u_h, \varphi_h) = 0, & \text{on } \Gamma(t), \\ u_h(0) = w_h, & \text{on } \Gamma(0) \end{cases} \quad (48) \text{ ?eq:20?}$$

The problems (48) and (47) differ in the following way: If the initial data for (47) is constant then it remains so for all times. In general this does not hold for solutions of (48). On the other hand (48) preserves the mean value of its initial data, which is in general not true for a solution of (47).

Proof of Lemma 5.1. First we investigate the finite element matrix representation of $E_h(t, s)$ with respect to the standard finite element basis, which we denote by $\mathbf{E}_h(t, s)$. From (45) we have

$$\frac{d}{dt} (M(t) \mathbf{E}_h(t, 0) \mathbf{u}_h(0)) + A(t) \mathbf{E}_h(t, 0) \mathbf{u}_h(0) = 0.$$

Let $\Lambda(t, s)$ the resolvent matrix of the ODE

$$\frac{d\xi}{dt} + A(t) M(t)^{-1} \xi = 0.$$

Then obviously it holds

$$\mathbf{E}_h(t, s) = M(t)^{-1} \Lambda(t, s) M(s).$$

Denote by $\mathbf{E}_h(t, s)^*$ the matrix representation of $E_h(t, s)^*$. From equation (46) it follows

$$\mathbf{E}_h(t, s)^* = M(s)^{-1} \mathbf{E}_h(t, s)^T M(t) = \Lambda(t, s)^T.$$

Now we calculate $\frac{d\Lambda(t, s)}{ds}$. Note that $\Lambda(t, s) = \Lambda(s, t)^{-1}$ and it holds

$$\frac{d\Lambda(s, t)^{-1}}{ds} = -\Lambda(s, t)^{-1} \frac{d\Lambda(s, t)}{ds} \Lambda(s, t)^{-1}.$$

From that it easily follows

$$\frac{d\Lambda(t, s)}{ds} = \Lambda(t, s) A(s) M(s)^{-1},$$

which now implies

$$\frac{d\mathbf{E}_h(t, s)^*}{ds} = M(s)^{-1} A(s) \mathbf{E}_h(t, s)^*.$$

□

5.2 A discrete delta and Green's function

Let $\delta_h = \delta_h^{x_h} = \delta_h^{t, x_h} \in S_h(t)$ be a finite element discrete delta function defined as

$$m_h(t; \delta_h^{t, x_h}, \varphi_h) = \varphi_h(x_h, t) \quad (\varphi_h \in S_h(t)). \quad (49) \text{ ?eq:15?}$$

If $\delta^{x_h} : \Gamma_h(t) \rightarrow \mathbb{R}$ is a smooth function having support in the triangle E_h containing x_h , then since $\dim \Gamma_h(t) = 2$ one easily calculates $\|\delta^{x_h} \sigma^{x_h}\|_{L^2(\Gamma_h(t))} \leq c$ for some constant independent of h and t . For the discrete delta function δ_h a similar result holds.

Lemma 5.2. *There exists $c > 0$ independent of t and h :*

$$\|\sigma^{x_h} \delta_h^{x_h}\|_{L^2(\Gamma_h(t))} \leq c \quad (x_h \in \Gamma_h(t)).$$

The proof is a straight forward extension of the corresponding one in Schatz, Thomée, Wahlbin [25] and uses the exponential decay property of the L^2 -projection, cf. Theorem 3.1 (24).

Next we define a finite element discrete Green's function as follows. Let $s \in [0, T]$. For given $u_h \in S_h(s)$ there exists a unique $\psi_h \in S_h(s)$ such that

$$a_h^*(s; \psi_h, \varphi_h) = m_h(s; u_h, \varphi_h) \quad \forall \varphi_h \in S_h(s).$$

This defines an operator

$$T_h^{*,s} : S_h(s) \rightarrow S_h(s), \quad T_h^{*,s} u_h := \psi_h.$$

We call $G_h^{s,x} := T_h^{*,s} \delta_h^{s,x}$ a discrete Green's function.

A short calculation shows that for all $0 \neq \varphi_h \in S_h(s)$ it holds

$$m_h(s; T_h^{*,s} \varphi_h, \varphi_h) > 0,$$

which implies that $G_h^{s,x}(x) > 0$. Actually we can bound the singularity x with $c|\log h|$.

Lemma 5.3. *For the discrete Green's function $G_h^{s,x}$ we have the estimate*

$$G_h^{s,x}(x) \leq c|\log h|.$$

Proof. Using Lemma 3.12 with (4) we estimate as

$$\|G_h^{s,x}\|_{L^\infty(\Gamma_h(s))} \leq c|\log h|^{1/2} \|G_h^{s,x}\|_{H^1(\Gamma_h(s))} = c|\log h|^{1/2} \sqrt{G_h^{s,x}(x)}.$$

□

The next lemma needs a different treatment then the one presented in Schatz, Thomée and Wahlbin [25]. The reason for that is that the mass and stiffness matrix depend on time and further the stiffness matrix is singular.

Lemma 5.4. *Let u_h a solution of (47). Then we have the estimate*

$$\int_0^t \|u_h\|_{L^2(\Gamma_h(s))}^2 ds \leq c \cdot m_h(t; T_h^{*,t} u_h, u_h).$$

Proof. Note that Lemma 3.2 (8) reads with the matrix notation as follows: If \mathbf{Z}_h and $\boldsymbol{\phi}_h$ are the coefficient vectors of some finite element function, then we have the estimate

$$\begin{aligned}\mathbf{Z}_h^T \frac{dM(s)}{ds} \boldsymbol{\phi}_h &\leq c \sqrt{\mathbf{Z}_h^T M(s) \mathbf{Z}_h} \sqrt{\boldsymbol{\phi}_h^T M(s) \boldsymbol{\phi}_h}, \\ \mathbf{Z}_h^T \frac{dA(s)}{ds} \boldsymbol{\phi}_h &\leq c \sqrt{\mathbf{Z}_h^T A(s) \mathbf{Z}_h} \sqrt{\boldsymbol{\phi}_h^T A(s) \boldsymbol{\phi}_h}.\end{aligned}\tag{50} \text{?eq:34?}$$

In the following with drop the s dependency. Let \mathbf{u} be the time dependent coefficient vector of u_h . Then we have

$$0 = -M \frac{d\mathbf{u}}{ds} + A\mathbf{u} = -M \frac{d\mathbf{u}}{ds} + (A + M)\mathbf{u} - M\mathbf{u}.$$

Equivalently we write this equation as

$$\begin{aligned}-\frac{1}{2} \frac{d}{ds} [\mathbf{u}^T M (A + M)^{-1} M \mathbf{u}] \\ = -\mathbf{u}^T M \mathbf{u} + \mathbf{u}^T M (A + M)^{-1} M \mathbf{u} - \frac{1}{2} \mathbf{u}^T \frac{d}{ds} [M (A + M)^{-1} M] \mathbf{u}.\end{aligned}$$

The last term expanded reads

$$\begin{aligned}\frac{1}{2} \mathbf{u}^T \frac{d}{ds} [M (A + M)^{-1} M] \mathbf{u} \\ = \mathbf{u}^T \frac{dM}{ds} (A + M)^{-1} M \mathbf{u} + \frac{1}{2} \mathbf{u}^T M \frac{d(A + M)^{-1}}{ds} M \mathbf{u} = I_1 + I_2.\end{aligned}$$

Using (50) and a Young inequality we estimate as

$$\begin{aligned}|I_1| &\leq c \cdot \mathbf{u}^T M (A + M)^{-1} M \mathbf{u} + \frac{1}{2} \mathbf{u}^T M \mathbf{u}. \\ |I_2| &= \frac{1}{2} \left| \mathbf{u}^T M (A + M)^{-1} \frac{d(A + M)}{ds} (A + M)^{-1} M \mathbf{u} \right| \\ &\leq c \cdot \mathbf{u}^T M (A + M)^{-1} M \mathbf{u}\end{aligned}$$

Putting everything together we reach at

$$-\frac{d}{ds} [\mathbf{u}^T M (A + M)^{-1} M \mathbf{u}] \leq -\mathbf{u}^T M \mathbf{u} + c \cdot \mathbf{u}^T M (A + M)^{-1} M \mathbf{u}.$$

The claim then follows from Lemma C.1. \square

5.3 A weak discrete maximum principle

Proposition 5.2. *Let $U_h(x, t) \in S_h(t)$ the ESFEM solution of our linear heat problem. Then there exists a constant $c = c(T, v) > 0$, which depends exponentially on T and v such that*

$$\|U_h(t)\|_{L^\infty(\Gamma_h(t))} \leq c |\log h| \|U_h(0)\|_{L^\infty(\Gamma_h(0))}.$$

Proof. There exists $x_h \in \Gamma_h(t)$ such that

$$\begin{aligned}\|U_h(t)\|_{L^\infty} &= |U_h(x_h, t)| = m_h(t; U_h(t), \delta_h^{t, x_h}) = m_h(t; E(t, 0)U_h^0, \delta_h^{t, x_h}) \\ &= m_h(0; U_h^0, E(t, 0)^* \delta_h^{t, x_h}) \leq \|U_h^0\|_{L^\infty} \|E(t, 0)^* \delta_h^{t, x_h}\|_{L^1}.\end{aligned}$$

The claim follows from Lemma 5.5. \square

Lemma 5.5. For $G_h^x(t, s) := E_h(t, s)^* \delta_h^{t, x}$, where $\delta_h^{t, x}$ is defined via (49) and $E_h(t, s)^*$ is defined via (46), it holds

$$\|G^x(t, 0)\|_{L^1(\Gamma_h(0))} \leq c|\log h|,$$

where the constant $c = c(T, v)$ depending exponentially on T and v such and is independent of x, h, t and s .

Proof. The proof presented here is a modification of the proof from Schatz, Thomée and Wahlbin [25, Lemma 2.1]. We estimate

$$\|G_h^x(t, 0)\|_{L^1(\Gamma_h(0))} \leq \|1/\sigma^x\|_{L^2(\Gamma_h(0))} \|\sigma^x G_h^x(t, 0)\|_{L^2(\Gamma_h(0))}.$$

With subsection B.1 it follows

$$\|1/\sigma^x\|_{L^2(\Gamma_h(0))}^2 \leq c|\log h|.$$

It remains to show

$$\|\sigma^x G_h^x(t, 0)\|_{L^2(\Gamma_h(0))}^2 \leq c|\log h|.$$

In the following we abbreviate $\sigma = \sigma^x$ and $G_h = G_h^x(t, s)$. With equation (47) and the discrete transport property we proceed as follows

$$\begin{aligned}& -\frac{1}{2} \frac{d}{ds} \|\sigma G_h\|_{L^2(\Gamma_h(s))}^2 + \|\sigma \nabla_{\Gamma_h} G_h\|_{L^2(\Gamma_h(s))}^2 \\ &= -m_h(s; \partial_h^{\bullet, s} G_h, \sigma^2 G_h) + a_h(s; G_h, \sigma^2 G_h) \\ &\quad - 2m_h(s; \sigma \nabla_{\Gamma_h} G_h, G_h \nabla_{\Gamma_h} \sigma) \\ &\quad - m_h(s; \partial_h^{\bullet, s} \sigma, \sigma G_h^2) - \frac{1}{2} m_h(s; \sigma^2 G_h^2, \nabla_{\Gamma_h} \cdot V_h) \\ &= -m_h(s; \partial_h^{\bullet, s} G_h, \sigma^2 G_h - \psi_h) + a_h(s; G_h, \sigma^2 G_h - \psi_h) \\ &\quad - 2m_h(s; \sigma \nabla_{\Gamma_h} G_h, G_h \nabla_{\Gamma_h} \sigma) \\ &\quad - m_h(s; G_h \partial_h^{\bullet, s} \sigma, \sigma G_h) - \frac{1}{2} m_h(s; \sigma^2 G_h^2, \nabla_{\Gamma_h} \cdot V_h) \\ &= I_1 + I_2 + I_3 + I_4 + I_5.\end{aligned}$$

For the choice $\psi_h = P_0(\sigma^2 G_h)$ we have $I_1 = 0$. Using Cauchy–Schwarz inequality, Lemma 3.14 and an inverse estimate 3.11 we get

$$|I_2| \leq c(\|G_h\|_{L^2(\Gamma_h(s))}^2 + \|G_h\|_{L^2(\Gamma_h(s))} \cdot \|\sigma \nabla_{\Gamma_h} G_h\|_{L^2(\Gamma_h(s))}).$$

Using Lemma 3.13 (27) we reach at

$$|I_3| \leq c\|G_h\|_{L^2(\Gamma_h(s))} \|\sigma \nabla_{\Gamma_h} G_h\|_{L^2(\Gamma_h(s))}.$$

Using Lemma 3.13 (26) we have

$$\begin{aligned} |I_4| &\leq c \|G_h\|_{L^2(\Gamma_h(s))} \|\sigma G_h\|_{L^2(\Gamma_h(s))}, \\ |I_5| &\leq c \|\sigma G_h\|_{L^2(\Gamma_h(s))}^2. \end{aligned}$$

After a Young inequality we have

$$-\frac{d}{ds} \|\sigma G_h\|_{L^2(\Gamma_h(s))}^2 + \|\sigma \nabla_{\Gamma_h} G_h\|_{L^2(\Gamma_h(s))}^2 \leq c \|G_h\|_{L^2(\Gamma_h(s))}^2 + c \|\sigma G_h\|_{L^2(\Gamma_h(s))}^2.$$

Lemma C.1 yields

$$\|\sigma G_h(t, 0)\|_{L^2(\Gamma_h(0))}^2 \leq c \left(\int_0^t \|G_h(t, s)\|_{L^2(\Gamma_h(s))}^2 ds + \|\sigma^x \delta_h^x\|_{L^2(\Gamma_h(0))}^2 \right).$$

For the first term we get from Lemma 5.4 and Lemma 5.3 the bound

$$\int_0^t \|G_h(t, s)\|_{L^2(\Gamma_h(s))}^2 ds \leq c |\log h|.$$

The last term is bounded according to Lemma 5.2. \square

6 Convergence of the semidiscretization

Theorem 6.1. *Let $\Gamma(t)$ be an evolving surface, let $u: \Gamma(t) \rightarrow \mathbb{R}$ be the solution of (1) and let $u_h = U_h^l \in H^1(\Gamma(t))$ be the solution of (3). If it holds*

$$\|P_{h,1}(t)u - U_h\|_{L^\infty(\Gamma_h(t))} \leq ch^2,$$

then there exists $h_0 > 0$ sufficiently small and $c = c(h_0) > 0$ independent of t , such that for all $0 < h < h_0$ we have the estimate

$$\begin{aligned} \|u - u_h\|_{L^\infty(\Gamma(t))} + h \|u - u_h\|_{W^{1,\infty}(\Gamma(t))} \\ \leq ch^2 |\log h|^4 (1+t) (\|u\|_{W^{2,\infty}(\Gamma(t))} + \|\partial^\bullet u\|_{W^{2,\infty}(\Gamma(t))}). \end{aligned}$$

Proof. It suffices to prove the L^∞ -estimate, since an inverse inequality implies the $W^{1,\infty}$ -estimate.

For this proof we denote by $P_{h,1}u = P_{h,1}(t)u$, $P_1u = (P_{h,1}u)^l$ and $u_h = U_h^l$. We split the error as follows

$$u - u_h = (u - P_1u) + (P_{h,1}u - U_h)^l = \sigma + \theta_h^l.$$

Because of Theorem 4.1 it remains to bound θ_h . Obviously there exists $R_h \in S_h(t)$ such that for all $\phi_h \in S_h(t)$ it holds

$$\frac{d}{dt} \int_{\Gamma_h(t)} \theta_h \phi_h + \int_{\Gamma_h(t)} \nabla_{\Gamma_h} \theta_h \cdot \nabla_{\Gamma_h} \phi_h - \int_{\Gamma_h(t)} \theta_h \partial_h^\bullet \phi_h = \int_{\Gamma_h(t)} R_h \phi_h.$$

By the variation of constant formula we deduce

$$\theta_h(t) = E_h(t, 0) \theta_h(0) + \int_0^t E_h(t, s) R_h(s) ds.$$

With Proposition 5.2 we get

$$\|\theta_h\|_{L^\infty(\Gamma_h(t))} \leq c|\log h|(\|\theta_h(0)\|_{L^2(\Gamma_h(t))} + t \max_{s \in [0,t]} \|R_h(s)\|_{L^\infty(\Gamma_h(t))}).$$

Observe that if we denote by $\varphi_h := \phi_h^l$, then a quick calculation reveals

$$\begin{aligned} m_h(R_h, \phi_h) &= m_h(\partial_h^\bullet P_{h,1}u, \phi_h) + g_h(V_h; P_{h,1}u, \phi_h) + a_h(P_{h,1}u, \phi_h) \\ &\quad - (m(\partial_h^\bullet u, \varphi_h) + g(v_h; u, \varphi_h) + a(u, \varphi_h)) \end{aligned} \quad (51) \text{ ?eq:54?}$$

Lemma 6.1 finishes the proof. \square

Lemma 6.1. *Assume that $R_h \in S_h(t)$ satisfies for all $\phi_h \in S_h(t)$ with $\varphi_h := \phi_h^l$ equation (51). Then it holds*

$$\|R_h\|_{L^\infty(\Gamma_h(t))} \leq ch^2|\log h|^3(\|u\|_{W^{2,\infty}(\Gamma(t))} + \|\partial^\bullet u\|_{W^{2,\infty}(\Gamma(t))}).$$

Proof. Using Definition 3.3 (23), (51) and since L^∞ is the dual of L^1 we deduce

$$\|R_h\|_{L^\infty(\Gamma_h(t))} = \sup_{\substack{f_h \in L^1(\Gamma_h(t)) \\ \|f_h\|_{L^1(\Gamma_h(t))}=1}} m_h(R_h, f_h) = \sup_{\substack{f_h \in L^1(\Gamma_h(t)) \\ \|f_h\|_{L^1(\Gamma_h(t))}=1}} m_h(R_h, P_0 f_h).$$

Now consider

$$\begin{aligned} m_h(R_h, P_0 f_h) &= m_h(\partial_h^\bullet P_{h,1}u, P_0 f_h) - m(\partial_h^\bullet u, P_0 f_h^l) \\ &\quad + g_h(V_h; P_{h,1}u, P_0 f_h) - g(v_h; u, P_0 f_h^l) \\ &\quad + a_h(P_{h,1}u, P_0 f_h) - a(u, P_0 f_h^l) \\ &= I_1 + I_2 + I_3. \end{aligned}$$

Using Lemma 3.3 and Theorem 3.1 it is easy to see

$$\begin{aligned} |I_1| &\leq c(\|\partial_h^\bullet u - \partial_h^\bullet(P_{h,1}u)^l\|_{L^\infty(\Gamma(t))} \\ &\quad + h^2(\|\partial^\bullet u\|_{L^\infty(\Gamma(t))} + h^2\|u\|_{W^{1,\infty}(\Gamma(t))}))\|f_h\|_{L^1(\Gamma_h(t))} \\ |I_2| &\leq c(\|u - P_{h,1}u^l\|_{L^\infty(\Gamma(t))} + h^2\|u\|_{L^\infty(\Gamma(t))})\|f_h\|_{L^1(\Gamma_h(t))} \\ |I_3| &\leq c(h^2\|u\|_{L^\infty(\Gamma(t))} + \|u - (P_{h,1}u)^l\|_{L^\infty(\Gamma(t))})\|f_h\|_{L^1(\Gamma_h(t))} \end{aligned}$$

Theorem 4.1 and Theorem 4.2 imply the claim. \square

7 A numerical experiment

We present a numerical experiment for an evolving surface parabolic problem discretized in space by the evolving surface finite element method. As a time discretization method we choose backward difference formula 4 with a sufficiently small time step (in all the experiments we choose $\tau = 0.001$).

As initial surface Γ_0 we choose the unit sphere $S^2 \subset \mathbb{R}^3$. The dynamical system is given by $\Phi(x, y, z, t) = (\sqrt{1 + 0.25 \sin(2\pi t)}x, y, z)$, which implies the velocity $v(x, y, z, t) = (\pi \cos(2\pi t)/(4 + \sin(2\pi t))x, 0, 0)$, over the time interval $[0, 1]$. As the exact solution we choose $u(x, y, z, t) = xye^{-6t}$. The complicated right-hand side was calculated using the computer algebra system Sage [6].

We give the errors in the following norm and seminorm

$$L^\infty(L^\infty) : \quad \max_{1 \leq n \leq N} \|u_h^n - u(\cdot, t_n)\|_{L^\infty(\Gamma(t_n))},$$

$$L^2(W^{1,\infty}) : \quad \left(\tau \sum_{n=1}^N |\nabla_{\Gamma(t_n)}(u_h^n - u(\cdot, t_n))|_{L^\infty(\Gamma(t_n))}^2 \right)^{1/2}.$$

The experimental order of convergence (EOC) is given as

$$EOC_k = \frac{\ln(e_k/e_{k-1})}{\ln(2)}, \quad (k = 2, 3, \dots, n),$$

where e_k denotes the error of the k -th level.

level	dof	$L^\infty(L^\infty)$	EOCs	$L^2(W^{1,\infty})$	EOCs
1	126	0.00918195	-	0.01921707	-
2	516	0.00308305	1.57	0.01481673	0.37
3	2070	0.00100752	1.61	0.00851267	0.80
4	8208	0.00025326	1.99	0.00399371	1.09

Table 1: Errors and EOCs in the $L^\infty(L^\infty)$ and $L^2(W^{1,\infty})$ norms

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A A Green’s function for evolving surfaces

Aubin [1, Section 4.2] proves existence of a Green’s function on a closed manifold M , that is a function which satisfies in $M \times M$

$$\Delta_Q \text{ distr. } \mathbf{G}(P, Q) = \delta_P(Q),$$

where Δ is the Laplace–Beltrami operator on M . The Green’s function is unique up to an constant. For Lemma 3.12 we need that the first derivative of a Green’s function can be bounded independent of t .

Theorem A.1 (Green’s function). *Let $\Gamma(t)$ with $t \in [0, T]$ be an evolving surface. There exists a Green’s function $\mathbf{G}(t; x, y)$ for $\Gamma(t)$. The value of $\mathbf{G}(x, y)$ depends only on the value of $\text{dist}_{\Gamma(t)}(x, y)$. $\mathbf{G}(x, y)$ satisfies the inequality*

$$|\nabla_{\Gamma}^x \mathbf{G}(t; x, y)| \leq c \frac{1}{\text{dist}_{\Gamma(t)}(x, y)}.$$

for some $c > 0$ independent of t .

Furthermore for all functions $\varphi \in C^2(\mathcal{G}_T)$ it holds

$$\varphi(x, t) = \frac{1}{V} \int_{\Gamma(t)} \varphi(y, t) dy - \int_{\Gamma(t)} \mathbf{G}(t; x, y) \Delta_{\Gamma} \varphi(y, t) dy. \quad (52) \text{ ?eq:8?}$$

Proof. As noted in Aubin [1, 4.10] the distance $r = \text{dist}_{\Gamma(t)}(x, y)$ is only a Lipschitzian function on $\Gamma(t)$. To use his construction we therefore need to revise that the injectivity radius at any point $P \in \Gamma(t)$ can be bounded by below by a number independent of P and t . This follows if the Riemannian exponential map is continuous in t and from Lemma C.2. To prove that the Riemannian exponential map is continuous one carefully revises the construction of exponential map as it is given in Chavel [4, Chapter 1]. Formula (52) follows from Aubin [1, Theorem 4.13] and that the constant is independent of t is a straightforward calculation. \square

B Calculations with some weight functions on evolving surfaces

B.1 Integration with geodesic polar coordinates on evolving surfaces

Assume we have sufficiently smooth function $f: \Gamma(t) \times \Gamma(t) \rightarrow \mathbb{R}$, where the value $f(x, y)$ depends on the distance $r = \text{dist}_{\Gamma(t)}(x, y)$ and we want to estimate the quantity $\int_{\Gamma(t)} f(x, y) dy$ for a fix y .

Applying the well known coarea formulae to the distance function r , cf. Chavel [4, Theorem 3.13] and Morgan [23, Theorem 3.13], we reach at

$$\begin{aligned} \int_{\Gamma(t)} f(x, y) dy &= \int_0^\infty \int_{\{\text{dist}_{\Gamma(t)}(x, y)=r\}} f(r) d\omega dr \\ &= \int_0^\infty \frac{\mathcal{H}^m(\{\text{dist}_{\Gamma(t)}(x, y) = r\})}{r^m} f(r) r^m dr, \end{aligned}$$

where \mathcal{H}^m denotes the m -dimensional Hausdorff measure. If $\Gamma(t)$ is not a closed surface but \mathbb{R}^{m+1} then $\frac{\mathcal{H}^m(\{\text{dist}_{\Gamma(t)}(x, y)=r\})}{r^m}$ would be constant. For closed surfaces the situation is different. Obviously there exists a positive number $R > 0$ independent of t and $x, y \in \Gamma(t)$ such that for all $r \geq R$ it holds

$$\mathcal{H}^m(\text{dist}_{\Gamma(t)}(x, y) = r) = 0.$$

Lemma B.1. *There exists $c > 0$ depending on $t \in [0, T]$ and $x \in \Gamma(t)$ such that*

$$\sup_{r>0} \frac{\mathcal{H}^m(\{\text{dist}_{\Gamma(t)}(x, y) = r\})}{r^m} \leq c.$$

Proof. It holds

$$\lim_{r \rightarrow 0} \frac{\mathcal{H}^m(\{\text{dist}_{\Gamma(t)}(x, y) = r\})}{r^m} = \omega_m,$$

where ω_m is the volume of the m -dimensional sphere in \mathbb{R}^{m+1} , cf. Gray [17, Theorem 3.1.]. Thus the proof is finished if find a $c > 0$ such that

$$\mathcal{H}^m(\{\text{dist}_{\Gamma(t)}(x, y) = r\}) \leq c \quad \forall r \in [0, \infty).$$

This can be seen as follows. For a fix point $p \in \Gamma(t)$ it is possible to use the Riemannian exponential to flat out $\Gamma(t)$, cf. figure ?? for an illustration on the torus. We make this argument precise.

For $r \in [0, \infty)$ let

$$S_p(r) := \{v \in T_p\Gamma(t) \mid g_p(v, v) = r^2\}$$

be the sphere of radius r and for $v \in S_p(1)$ consider the geodesic

$$f_v: [0, \infty) \rightarrow \Gamma, \quad \lambda \mapsto \exp_p(\lambda v).$$

It is well known that a geodesic is just locally length minimizing. Hence there exists a unique $\lambda_*(v) > 0$, such that $f_v|_{[0, \lambda_*(v)]}$ is a length minimizing geodesic and for every $\varepsilon > 0$ $f_v|_{[0, \lambda_*(v) + \varepsilon]}$ is not anymore length minimizing. We define

$$W_p(t) := \{w \in T_p\Gamma(t) \mid w = \lambda \cdot v \text{ with } v \in S_p \text{ and } \lambda \in [0, \lambda_*(v)]\}.$$

Obviously it holds for every $w \in W_p(t)$ that $\text{dist}_{\Gamma(t)}(p, \exp_p(w)) \leq R$. Further there exists for every $q \in \Gamma$ a unique $w \in W_p$ with $\exp_p(w) = q$. Clearly it holds

$$\exp_p(W_p \cap S_p(r)) = \{\text{dist}_{\Gamma(t)}(x, y) = r\}.$$

Now apply a general Area-coarea Formula, cf. [23, Theorem 3.13], to finish the proof. \square

Using this lemma we have the estimate

$$\int_{\Gamma(t)} f(x, y) dy \leq c \int_0^R r^m f(r) dr.$$

B.2 Comparison of extrinsic and intrinsic distance

Lemma B.2. *There exists a constant $c > 0$ independent of t such that for all $x, y \in \Gamma(t)$ the following inequality holds*

$$c \cdot \text{dist}_{\Gamma(t)}(x, y) \leq |x - y|. \quad (53) \text{ ?eq:3?}$$

Proof. For simplicity we assume that $\Gamma(t) = \Gamma_0$ for all $t \in [0, T]$. The basic idea is to find a radius $r > 0$ and two constant $c_1, c_2 > 0$ such that (53) holds with c_1 for $\text{dist}_{\Gamma(t)}(x, y) \leq r$ and with c_2 for $\text{dist}_{\Gamma(t)}(x, y) \geq r$.

Observe that from the compactness from Γ_0 it follows that there exists $r > 0$ such that for all $\text{dist}_{\Gamma(t)}(x, y) \leq r$ it holds

$$\nu(x) \cdot \nu(y) \geq \cos(\pi/6).$$

After rotation we may assume $x = 0$, $\nu(x) = e_{n+1}$ and that Γ_0 may be written as graph of a smooth function, that means that there exists $f: U(x) \rightarrow \mathbb{R}$ smooth with $U(x) \subset \mathbb{R}^n$ an open subset, such that $z = (z', w) \in \Gamma_0 \subset \mathbb{R}^m \times \mathbb{R}$ with $\text{dist}_{\Gamma(t)}(z, x) \leq r$ if and only if $z' \in U(x)$ and $w = f(z')$. For $x = (0, 0)$ and $y = (y', f(y'))$ consider the path $t \mapsto (ty', f(ty'))$. We calculate

$$\text{dist}_{\Gamma(t)}(x, y) \leq \int_0^1 \sqrt{1 + df_{ty'} y'} dt \leq \sqrt{1 + \|f\|_{W^{1,\infty}}^2} |y'| \leq \sqrt{1 + \|f\|_{W^{1,\infty}}^2} |y - x|.$$

Now the derivatives of f are bounded by $m \cdot \tan(\pi/6)$.

To get the existence of $c_2 > 0$ observe that $\text{dist}_{\Gamma(t)}$ is continuous and hence the set $\text{dist}_{\Gamma(t)}^{-1}\{r > 0\}$ is compact. On this set the function $|x - y|$ does not vanish and takes its maximum and minimum. \square

B.3 Weight functions

Definition B.1. *Let μ and $\tilde{\mu}$ be like (9) resp. (25). For given $\mu, \tilde{\mu}$ with curve $y = y(t)$, we define a curve $y_h = y_h(t) := y(t)^{-l} \in \Gamma_h(t)$. Now we define a weight function on the discrete surface*

$$\mu_h: \Gamma_h(t) \rightarrow \mathbb{R}, \quad \text{resp.} \quad \tilde{\mu}_h: \Gamma_h(t) \rightarrow \mathbb{R},$$

via the same formula like (9) resp. (25).

Lemma B.3. *There exists a constant $h_0 = h_0(\gamma) > 0$ sufficiently small and $c = c(h_0) > 0$ independent of t and h such that for all $0 < h < h_0$ it holds*

$$\frac{1}{c}\mu \leq \mu_h^l \leq c\mu, \quad (54) \text{ ?eq:41?}$$

$$\frac{1}{c}\tilde{\mu} \leq \tilde{\mu}_h^l \leq c\tilde{\mu}. \quad (55) \text{ ?eq:42?}$$

Proof. The main idea is to observe that we have the inequalities

$$\begin{aligned} |x^{-l} - y_h| &\leq 2d + |x - y|, \\ |x - y| &\leq 2d + |x^{-l} - y_h|, \end{aligned}$$

where $d = d(t) := \max_{x \in \Gamma(t)} \text{dist}_{\mathbb{R}^{n+1}}(x, \Gamma_h(t))$. □

C Modified analytic results for evolving surface problems

Lemma C.1 (modified Gronwall inequality). *Let $c > 0$ be a positive constant, let φ, ψ and ρ be some positive functions defined on $[t, T]$ and assume for all $s \in [t, T]$ we have the inequality*

$$-\frac{d\varphi}{ds}(s) + \psi(s) \leq c\varphi(s) + \rho(s).$$

Then it holds

$$\varphi(t) + \int_t^T \psi(s) ds \leq e^{c(T-t)} \left(\varphi(T) + \int_t^T \rho(s) ds \right).$$

Proof. Calculate $-\frac{d}{ds}[\varphi e^{-c(T-s)}]$ and integrate from t to T . □

Lemma C.2 (modified inverse function theorem). *Let $f: \mathbb{R}^n \times [0, T] \rightarrow \mathbb{R}^n$ be a smooth map, denote by $f(t)(x) := f(x, t)$ and assume that for all $t \in [0, T]$ the map $df(t)_0 = \frac{\partial f}{\partial x}(0, t)$ is invertible. Then there exists $r > 0$ independent of t such that*

$$f(t): f(t)^{-1}\{B_r(0)\} \rightarrow \mathbb{R}^n, \quad x \mapsto f(x, t),$$

is a diffeomorphism onto its image and we have $B_{r/2}(0) \subset f(t)^{-1}\{B_r(0)\}$ for all t , where $B_r(0) := \{x \in \mathbb{R}^n \mid |x| \leq r\}$. The map

$$g: [0, T] \times B_r(0) \rightarrow \mathbb{R}^n, \quad (t, x) \mapsto f(t)^{-1}(x)$$

is smooth. In particular g is smooth in t .

Proof. The results follows from the compactness of $[0, T]$ and the smoothness of f . □